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Experiments on transition in plane Couette flow

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The first flow visualization experimental results of transition in plane Couette flow are reported. The Couette flow water channel was of an infinite-belt type with counter-moving walls. The belt and channel walls were transparent making it possible to visualize the flow pattern in the streamwise–spanwise plane by utilizing fluid-suspended reflective flakes. Transition was triggered by a high-amplitude pointwise disturbance. The transitional Reynolds number, i.e. the lowest Reynolds number for which turbulence can be sustained, was determined to be 360 ± 10, based on half-channel height and half the velocity difference between the walls. For Reynolds numbers above this value a large enough amplitude of the initial disturbance gave rise to a growing turbulent spot. Its shape and spreading rate was determined for Reynolds numbers up to 1000.

1. Introduction

The stability of fluid flows, the transition to turbulence and the fully developed turbulent state have all been studied extensively since Reynolds (1883) made the first flow visualization study of laminar–turbulent transition in pipe flow over a century ago. Transition experiments (as well as numerical simulations) in wall-bounded flows have since then been performed in many different flow geometries such as boundary-layer and channel flows, water-table flows, but also in flows where other physical effects play an important role, such as centrifugal effects (e.g. curved channel flow) or system rotation effects (e.g. rotating channel flow). Two-dimensional linear theory is today, in many flow situations, a standard procedure to determine the lowest Reynolds number *Re* for which an infinitesimal disturbance can grow exponentially (we denote this the critical Reynolds number). The existence and growth of two-dimensional disturbances above the critical *Re* has been confirmed experimentally in, for instance, plane Poiseuille flow by Nishioka, Iida & Ichikawa (1975). Natural transition in plane Poiseuille flow, however, usually occurs at sub-critical Reynolds numbers through the formation of turbulent spots without first observing two-dimensional waves in the flow. Turbulent spots in this flow were first visualized by Carlson, Widnall & Peeters (1982) and later by Alavyoon, Henningson & Alfredsson (1986) who determined the transitional Reynolds number† to be 1100 (based on centreline velocity and half-channel height), whereas the critical Reynolds number is 5772 (Orszag 1971), i.e. in a 'noisy' environment natural transition may occur well below the critical *Re*. In analogy to this one finds in certain situations, for instance plane Couette flow (see figure 1) and pipe flow, that theory shows that linear

† By *transitional* Reynolds number we mean the lowest Reynolds number for which transition to turbulence occurs if a strong enough disturbance is applied, whereas the *critical* Reynolds number denotes the lowest Reynolds number for an exponentially growing infinitesimal disturbance.

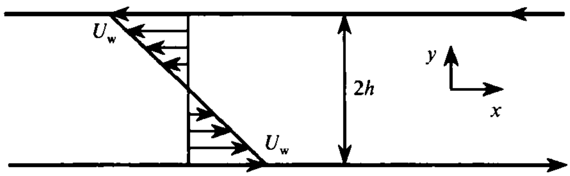


FIGURE 1. Geometry of plane Couette flow.

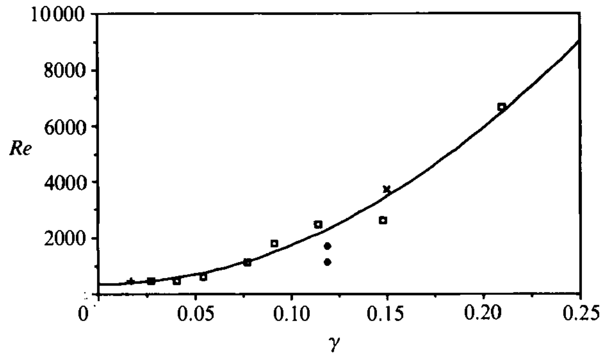


FIGURE 2. Transitional *Re*, found in cylindrical Couette flow with outer cylinder rotating, as a function of the ratio between channel width and radius of curvature: ◇, Coles (1965); +, Couette (1890); □, Taylor (1936); ×, Wendt (1933).

two-dimensional disturbances are damped, although experimentally transition to turbulence has been already observed at relatively low Reynolds numbers.

The present experimental study concerns transition in the conceptually most simple wall-bounded shear flow, namely the plane Couette flow. 'It seems reasonable that there is difficulty in performing experiments on plane Couette flow' is a citation from Ellingsen, Gjevik & Palm (1970) and this statement is supported by the fact that only a few experimental studies can be found in the literature. Hence knowledge derived from physical experiments on flow instability and transition to turbulence in plane Couette flow is limited. For instance the transitional Reynolds number (i.e. the Reynolds number for which turbulence may be self-sustained if the flow is triggered by a sufficiently large disturbance) is not accurately known, but falls, for various investigations, in the range of 280–750, where the Reynolds number of the flow is defined in terms of half the channel height (h) and half the velocity difference (U_w) between the two walls.

The experiment by Couette (1890) was done in order to determine the viscosity of water. He used an apparatus consisting of two coaxially mounted vertical circular cylinders with a narrow gap in between. The outer cylinder was rotating and was driving the flow, whereas the torque on the inner cylinder, which was stationary, was measured. From the torque on the inner cylinder the fluid viscosity could be determined, assuming that the flow was laminar and end effects were small. At a Reynolds number of around 480 an increase in the viscosity value was found, which can be ascribed to transition to turbulence. The value of the transitional *Re* in this geometry depends on the relation between the radius of the inner (R_i) and outer (R_o) cylinders or the ratio between the channel width and the radius of curvature $\{\gamma = (R_o - R_i)/R_o\}$. In the limit of $\gamma \rightarrow 0$ plane Couette flow is obtained; in Couette's experiment γ was as low as 0.015. In the experiments by Taylor (1936) γ was varied and the transitional *Re* was shown to increase with increasing γ, i.e. the centrifugal field has a stabilizing effect when rotating the outer cylinder (in contrast to rotating the inner cylinder, when Taylor vortices may form due to unstable stratification of the centrifugal force). Figure 2 shows the transitional *Re* of the flow between an inner stationary and an outer rotating cylinder from these investigations and those of Wendt (1933) and Coles (1965). It should be remarked that the higher value from Coles' study corresponds to the transitional *Re* as *Re* is monotonically increased from zero. The lower value is obtained as *Re* is decreased from a value where the flow is turbulent, i.e. there is a considerable hysteresis effect. An extrapolation of the data in figure 2 gives an estimate of the transitional *Re* in the region 300–500 as $\gamma \rightarrow 0$. From a somewhat more limited data base Wendt (1933) approximated the transitional *Re* to be 500 as $\gamma \rightarrow 0$.

Only few experiments in plane Couette flow concerning transition to turbulence can be found in the literature. Reichardt (1956) carried out measurements on both laminar and turbulent flow in a running belt apparatus and observed turbulence at *Re* of about 750. Leutheusser & Chu (1971, see also Aydin & Leutheusser 1979) carried out transition experiments in a facility where a free surface water flow was used as the moving wall. The other wall was a stationary flat plate located above and parallel with the free water surface, and air was drawn into the channel formed by the moving water surface and the stationary plate. The spanwise aspect ratio of their channel was small (12) and the water surface probably rough as they stated that the water flow was turbulent. The transitional *Re* determined by them was 280, but must be regarded with some caution owing to the shortcomings of the experimental set-up mentioned above. No flow visualization was reported of the transition process in these two experiments.

Two-dimensional Tollmien–Schlichting (TS) waves are damped in plane Couette flow for all Reynolds numbers (see e.g. Drazin & Reid 1981; Davey 1973). However, there is still the possibility that infinitesimal three-dimensional disturbances will grow through a coupling between vertical vorticity modes and TS-modes as suggested by Gustavsson & Hultgren (1980) and under more general conditions by Gustavsson (1991). This mechanism allows initial algebraic growth of infinitesimal disturbances at sub-critical Reynolds numbers, although the linear development will eventually lead to exponential decay of the disturbance. In an experimental situation the linear disturbance may, during the growth phase, reach a sufficiently high amplitude and become nonlinear, which can lead to further growth, breakdown and turbulent spot formation.

Attempts to obtain a transitional *Re* for plane Couette flow were made in the numerical simulation of Orszag & Kells (1980) and in the secondary instability analysis of Orszag & Patera (1983), which both showed that strong energy growth of a secondary instability occurs for a Reynolds number around 1000. The secondary instability analysis was carried out for a flow in which a three-dimensional disturbance was allowed to grow on a two-dimensional disturbance of moderate amplitude. However, no systematic search over the parameter space (i.e. the wavenumbers of the primary and secondary instabilities) was reported and the critical Reynolds number (i.e. 1000) should probably be viewed as an upper limit for the transitional *Re* in plane Couette flow.

In a recent numerical simulation by Lundbladh & Johansson (1991) it was shown that turbulent spot formation may occur in plane Couette flow if a high-amplitude localized disturbance is introduced in the flow. They carried out simulations at various Reynolds numbers and determined the transitional *Re* to be approximately 375. We will carry out comparisons between the present experimental findings and these numerical results later in this paper. However, already at this stage we may say as a general remark that the overall agreement is good.

In this paper we describe an experimental apparatus constructed to allow flow visualization of transition in plane Couette flow. We show that transition may occur through the formation of turbulent spots and we have determined the spreading of the turbulent region as a function of Reynolds number. The transitional Reynolds number has been determined to be 360 ± 10. In §2 the flow apparatus is described, and the experimental results are given and discussed in §§3 and 4.

2. Experimental apparatus

The experiments were done in a water channel of infinite-belt type, where for the experiments reported here both walls were moving at the same speed but in opposite directions. The running belt was made of a transparent plastic material ('overhead' film type) and allowed optical access to the channel for flow visualization. The flow was visualized by seeding the water with titanium-dioxide-coated platelets (Merck Irodin 120, 5–20 μm in diameter, of concentration less than 0.02% by volume) and illuminating the channel at an oblique angle (with a 150 W slide projector). The reflected light from the particles clearly revealed areas of turbulent and perturbed laminar flow.

The experimental apparatus is shown in figure 3, where the main parts are: the tank, the steel frame, the driving and steering cylinders and the plastic film formed into an infinite loop. The tank was made of 10 mm acrylic plastic, except the part that formed the channel which was made from a 10 mm thick floated glass plate (1). Its outer dimensions were $2.55 \times 0.45 \times 0.55$ m and the length of the test section (and hence the glass plate) was 1.50 m. To form the Couette channel a second glass plate of the same size was placed in the tank parallel to the glass plate forming part of the outer wall of the tank. The distance between the facing surfaces was adjusted by gauge blocks at the bottom and top and could be chosen freely within approximately 5 to 50 mm. In our experiments the gap was 10mm. To produce the plane Couette flow a transparent polyester plastic film (2) (0.1 mm thick and 360 mm wide) ran long the glass surfaces. The joint of the plastic film was made by gluing the film together with silicone glue. No effect of the joint could be observed in the flow visualization. The film was driven and supported by two precision made brass cylinder (3, 4) with a length of 450 mm and a diameter of 200 mm. The driving cylinder (3) was connected to a servo coupled DC motor (5). The second cylinder (4) controlled both the tension and the vertical position of the plastic film. An optical pulse generator (6) connected to the shaft measured the film speed. To ensure the right amount of lubricating fluid film between the glass surface and the plastic film, the position of the plastic film at the entrance of the Couette channel was controlled by a pair of freely rotating cylinders (7). Each cylinder could be accurately positioned by fine thread screws. A microscope was used to determine the exact position of the plastic film in the channel.

Controlled disturbances of short duration were made by a solenoid-activated, 1 mm diameter fluid jet (8). The jet entered the channel when a 3 mm diameter opening in the plastic belt coalesced with the jet orifice which is located in the centre of the inner wall. A photodetector controlled the timing of the solenoid and the start of a digital clock, which via a mirror arrangement showed up on the photographs (or video recordings) that were taken to document the flow development. The strength of the disturbance could be adjusted by varying the amount of fluid injected.

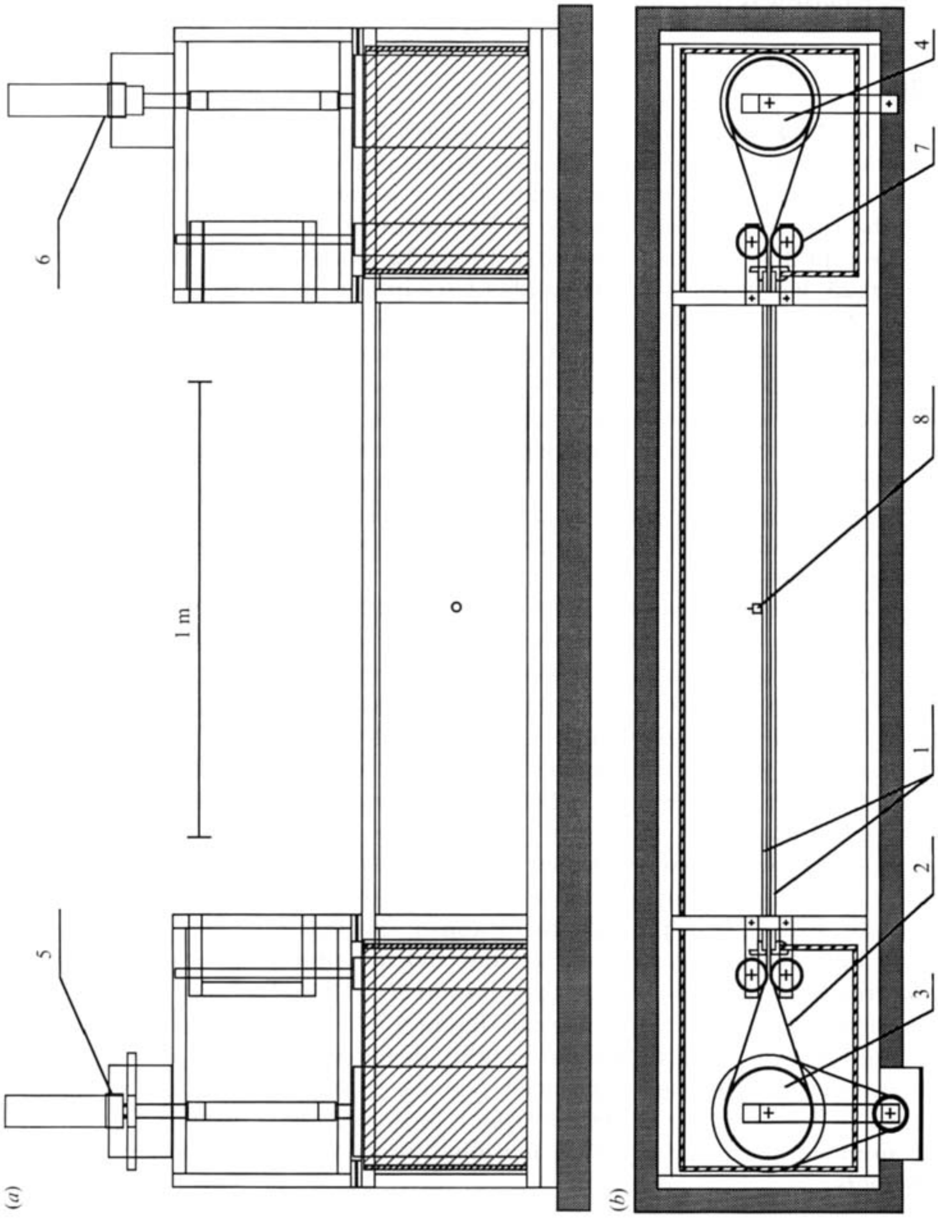


FIGURE 3. Experimental apparatus. 1: Floated glass plates, 2: polyester belt formed into an infinite loop, 3: driving cylinder, 4: belt supporting and speed measuring cylinder, 5: servo coupled DC-motor with gear box, 6: belt speed indicator, 7: belt steering cylinders, 8: liquid jet injector for spot triggering. (*a*) Side view, (*b*) top view.

LDV measurements with good signal quality were possible with a backscatter arrangement. In order to enable measurements of both positive and negative streamwise velocities, frequency shifting with a Bragg cell was used. Owing to the geometrical set-up the moving walls were normal to the major axes of the measuring volume. Its length was 0.3 mm, i.e. 3% of the channel width.

3. Results

After accelerating the walls to the desired speed, viscous diffusion will ultimately lead to the development of a linear velocity profile (see e.g. Batchelor 1967, p. 191). The timescale for this is the viscous scale h^2/ν (ν is the kinematic viscosity of the fluid) and the time in seconds is hence independent of the Reynolds number. The linear velocity profile is established to within 5% of the wall speed in a time of $\eta = t\nu/h^2 = 0.25$, which corresponds to about 6 s in our apparatus. Figure 4 shows examples of the velocity profile at two intermediate stages together with velocity measurements (LDV) at two *Re* for $\eta = 0.1$. The experiments confirm that the profile development is independent of Reynolds number. The major objective of the present study was to accurately determine the transitional Reynolds number for plane Couette flow. In the experiment the linear profile is reached asymptotically and one may be worried that the undeveloped flow would be unstable. To investigate this we have carried out Orr–Sommerfeld calculations with the theoretically determined developing profiles and for Reynolds numbers corresponding to our experiments no unstable modes were found.

In an experiment the start-up phase is unavoidable (cf. the inlet length of a channel or pipe flow experiment) and would not be a problem if the flow could be established without being subject to spontaneous transition. However, inlet/outlet disturbances in an experimental Couette flow apparatus with counter-moving walls are unavoidable and such disturbances propagate into the channel more or less as a straight front. This gives a limit to the time available for profile development and studies of controlled disturbances in the central part of the channel. The problem increases with the Reynolds number of the flow, since the time for establishment of the linear mean flow, scaled with the timescale relevant for the propagation of disturbances, i.e. h/U_w, is proportional to the Reynolds number. At low *Re* (less than approximately 400) it was possible to establish the linear profile without being troubled by inlet/outlet disturbances as they did not propagate into the channel. LDV-measurements of the fully developed flow at $Re = 345$ showed a linear velocity profile across the main part of the channel (figure 5). At high *Re*, on the other hand, it is not possible to wait until the profile is 'fully' developed to look at spot behaviour as the disturbances from the inlet/outlet regions do propagate into the channel. However, we believe that the deviation from a fully developed profile at the highest Reynolds numbers of this study only marginally affected the development of the turbulent spots.

If the flow is started from rest and the Reynolds number is increased by small increments the flow will finally become fully turbulent. Turbulence will start due to disturbances at the ends of the channel and will propagate into and eventually fill the channel. To determine the transitional *Re* in this way seems to be less satisfactory as inlet/outlet conditions play a major role. To avoid such effects a large disturbance was introduced in the centre of the channel and it was determined whether it developed into a turbulent region. The solenoid-driven jet was not a sufficiently large disturbance for this approach, but by introducing a large air bubble at the bottom

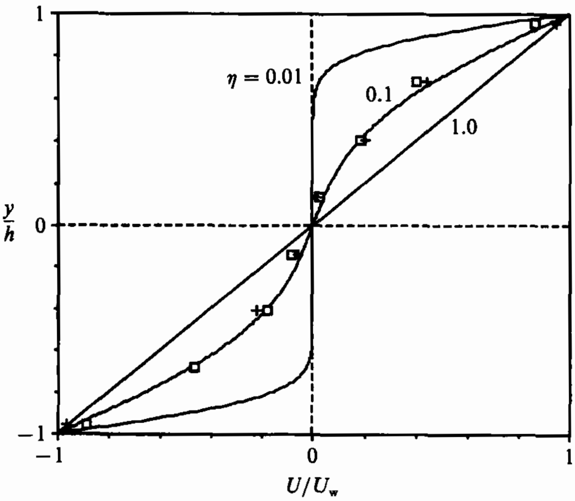


FIGURE 4. Velocity profiles for developing Couette flow. Theoretical profiles for $\eta = 0.01$, 0.1 and 1.0. Experimental data for $\eta = 0.1$: □, $Re = 365$; +, $Re = 650$.

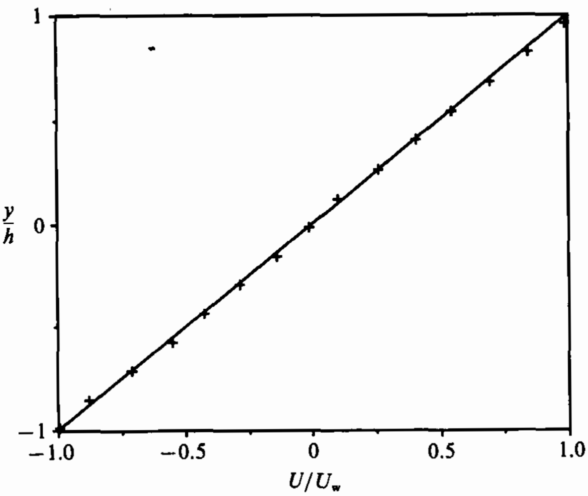


FIGURE 5. Measured velocity profile at $Re = 345$.

of the channel a strong disturbance was created when the bubble moved towards the surface. Another approach is to start with a turbulent state, i.e. at a high Reynolds number and then gradually decrease *Re* until all turbulence has disappeared. Both methods were tried and gave the same result for the transitional *Re*, namely 360 ± 10. At the transitional *Re* the turbulent region continuously changed its shape and it was impossible to tell whether a particular disturbance would survive or decay. In one case a turbulent region was observed for more than 20 minutes before it finally decayed and the flow became laminar. Note that, only by using an apparatus with counter-running belts is it possible in practice to follow the development of a turbulent region during such a long time. Long observation times were, however, necessary in order to accurately determine the transitional *Re*.

In the following photographic sequences the spot development at three different *Re*, namely 405, 722 and 912, is shown. (The reticulation seen in all photographs had

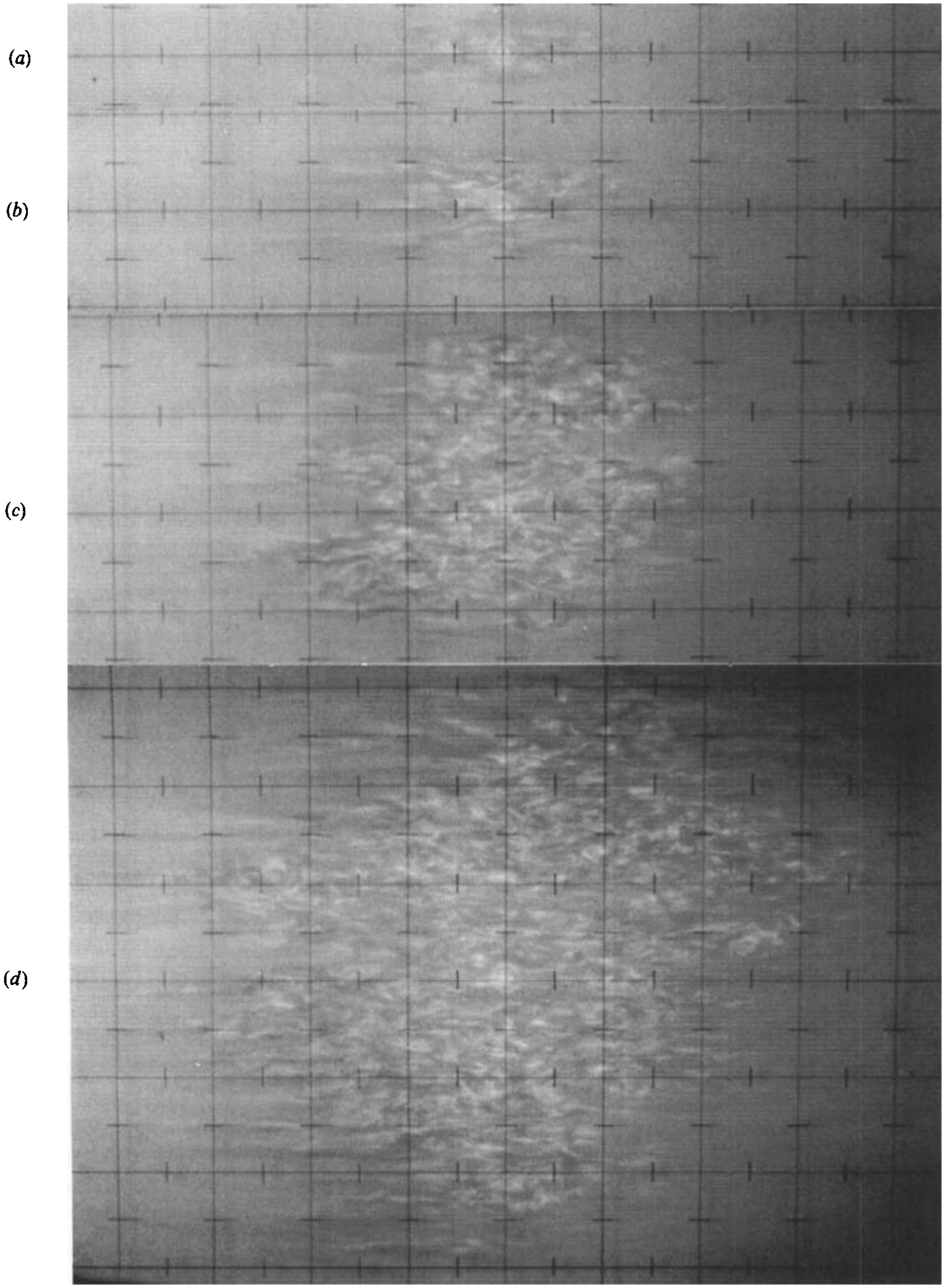
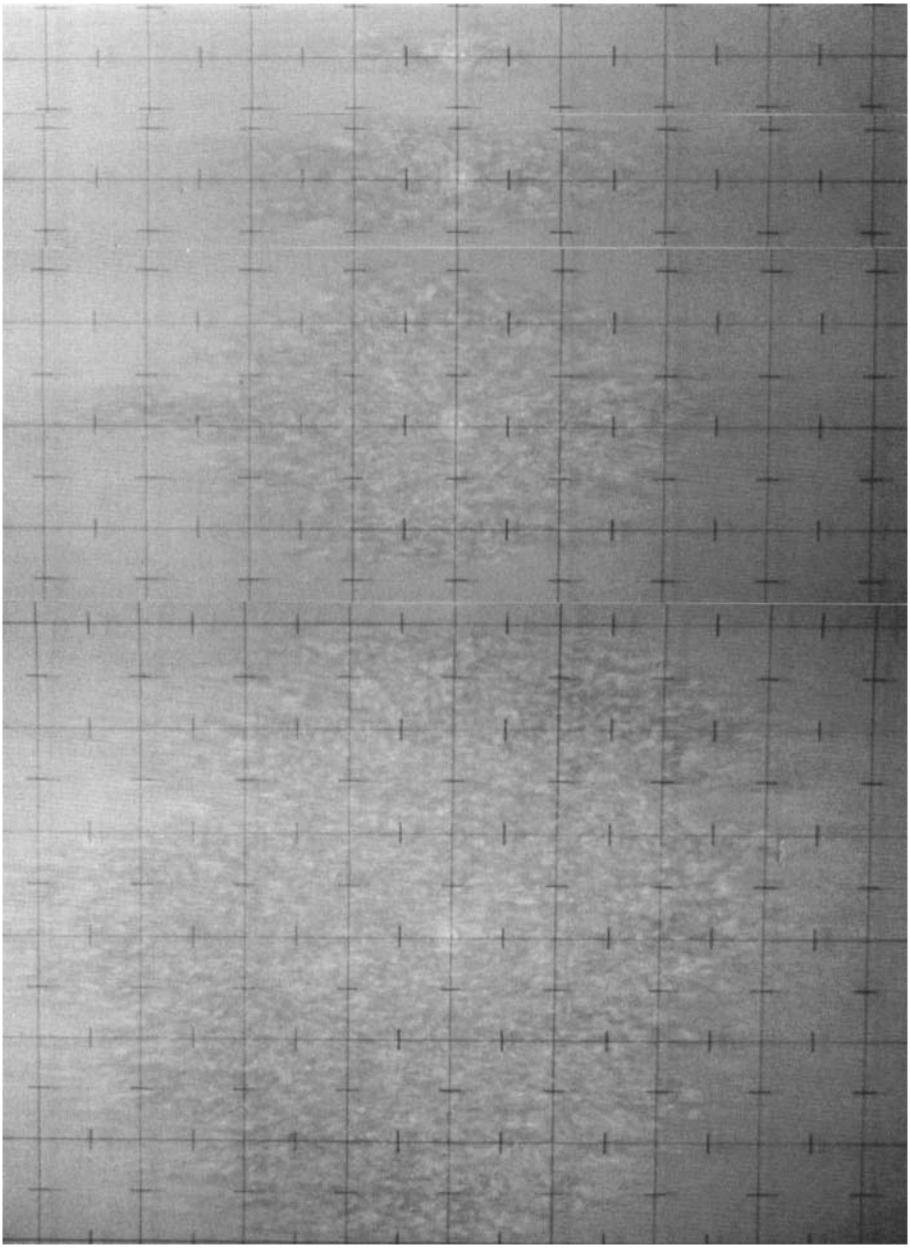


FIGURE 6. Turbulent spot development at $Re = 405$. Belt velocity is 89 mm/s. Non-dimensional time after triggering (a) 25, (b) 95, (c) 210, (d) 390.

a side length of 50 mm.) In all three cases the spots were triggered by the jet disturbance. At $Re = 405$, figure 6(a–d) shows a sequence of the spot from 1.4–20.5 s after triggering, corresponding to 25, 95, 210 and 390 in non-dimensional time units (t^*, scaled with belt speed and channel half-height). Initially a weak laminar disturbance was seen; however, in figure 6(b) breakdown has occurred in the upper



(*a*)
(*b*)
(*c*)
(*d*)

FIGURE 7. Turbulent spot development at $Re = 722$. Belt velocity is 158 mm/s. Non-dimensional time after triggering (*a*) 25, (*b*) 65, (*c*) 120, (*d*) 200.

part of the disturbed region. In figure 6(*c*) the turbulent region has grown and is approximately as wide as it is long. The boundaries are quite irregular with fingering turbulent patches stretching in the streamwise directions. One also notes that the turbulent scales seem to be fairly large. As time increased the spot became less symmetrical (figure 6*d*) and for even lower *Re* the turbulent region sometimes stretched as a diagonal band across the channel.

Figures 7(*a–d*) and 8(*a–d*) show the development of the spots at $Re = 722$ and 912, respectively. Figure 7(*a–d*) shows a sequence of the spot from 0.7–5.9 s ($t^* = 25$, 65,

(a)

(b)

(c)

(d)

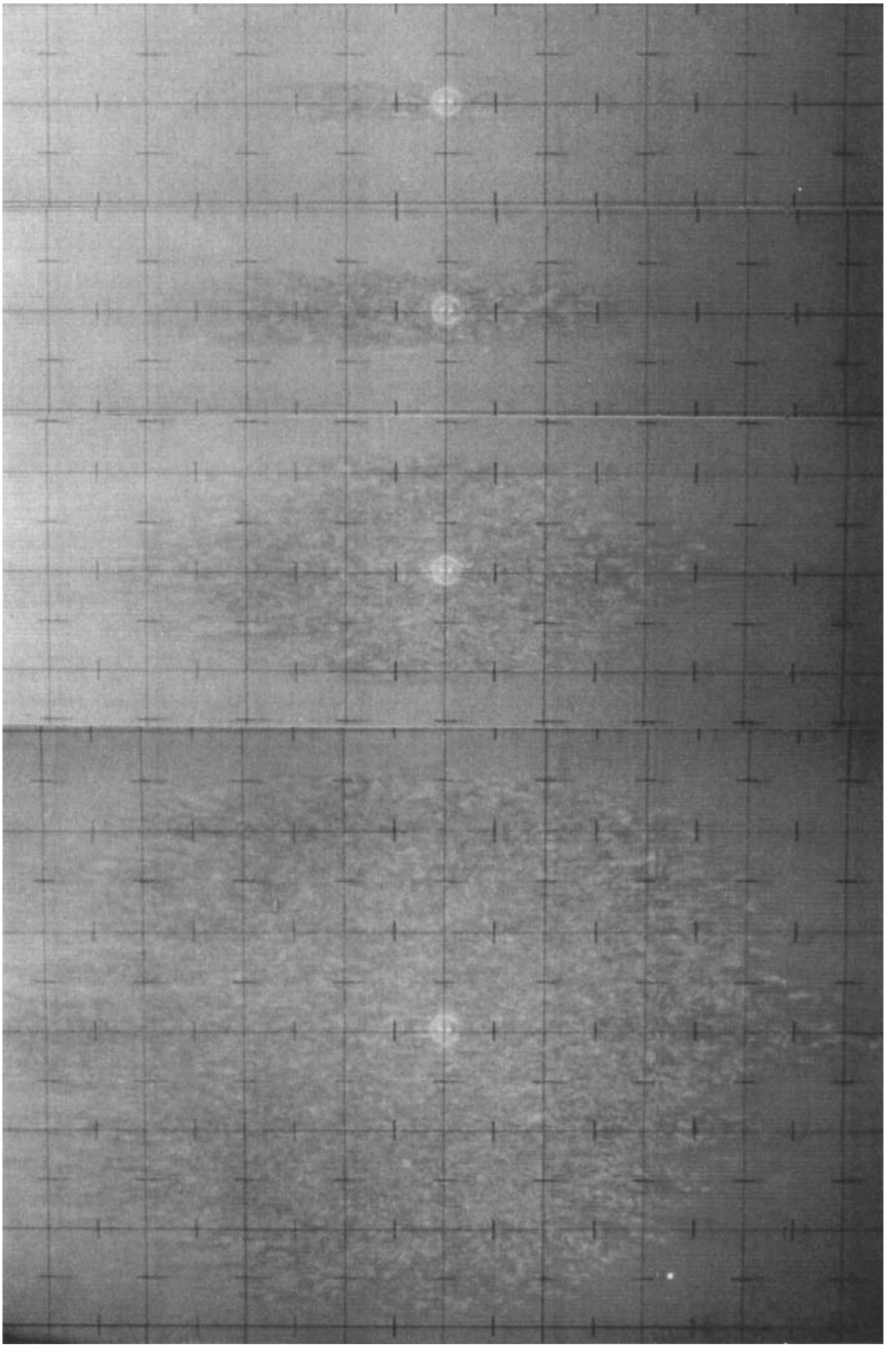


FIGURE 8. Turbulent spot development at $Re = 912$. Belt velocity is 200 mm/s. Non-dimensional time after triggering (a) 25, (b) 60, (c) 110, (d) 195.

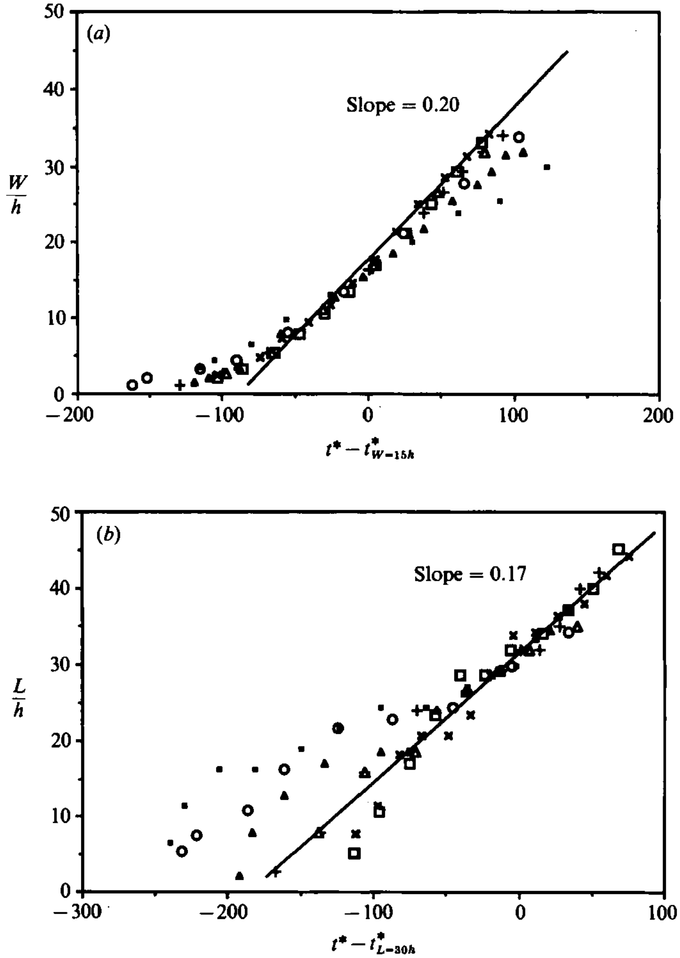


FIGURE 9. (*a*) Half-width and (*b*) half-length of seven different realizations as a function of time. Note that the time axis is shifted individually for each spot. □, $Re = 943$; ×, 832; +, 732; △, 634; ○, 545; ▲, 521; ■, 438.

120, 200), whereas figure 8(*a–d*) shows a sequence of the spot from 0.6–4.6 s ($t^* = 25$, 60, 110, 195).

For high Reynolds numbers the spot was initially more elongated than for low *Re*. When the spot grew older its aspect ratio (defined as its length-to-width ratio) decreased. This means that the spanwise growth was larger than the growth in the streamwise direction. At high *Re* the outer boundary of the spot was smoother and thereby more well-defined than at $Re = 405$ and the turbulence scales within the spot were smaller. For high Reynolds numbers the spots became quite symmetrical and they were found to grow in the streamwise and spanwise directions until the flow was fully turbulent.

Experiments were carried out at several Reynolds numbers and the half-length (L) and half-width (W) of the spots were determined as function of time (figure 9*a*, *b*). Here we have plotted the data for seven different *Re* (measurements at five other *Re* between 600 and 900 also fit the picture, but are left out for clarity) in one graph and shifted the time axis so that the data coincide when L and W for each spot were equal to approximately $30h$ and $15h$. This time is denoted by t^*_{L-30h} and t^*_{W-15h}, respectively. As seen from figure 9 the virtual time origin of the spots differs and is a consequence of the fact that the initial disturbances were the same for all *Re*, thereby becoming relatively stronger for the higher *Re*, leading to a faster initial growth. For higher *Re* all data in figure 9 coincide nicely for long enough times after triggering. The spread in the data for L was larger than for W, owing to the less distinct shape of the fronts/backs of the spot compared to the demarcation between the laminar and turbulent regions at the wing tip which was much clearer. The spreading rate was 0.17 in the streamwise and 0.20 in the spanwise direction. This means that, at least for short times, the spot will not reach a self-similar shape and eventually get an aspect ratio smaller than one. The spreading rate was constant at least for *Re* larger than 40% above the transitional *Re*, which is different from spots in plane Poiseuille flow where the spreading rate increases almost linearly with *Re* (Alavyoon *et al.* 1986). The half-spreading angle of the spot in plane Couette flow viewed in a coordinate system at rest relative to one of the belts is 11°, close to what is observed for Blasius boundary-layer spots (Gad-el-Hak, Blackwelder & Riley 1981).

Another feature of the flow which is not clearly seen in the photographs was the creation of waves at the wing tips of the spots. These waves were most pronounced at low *Re* and usually only one or two were observed. The wave crests were aligned with the streamwise direction (i.e. the wavenumber vector was perpendicular to the mean flow direction) and in the laboratory frame of reference they moved in the spanwise direction out from the spot. The waves showed up weakly in the flow visualization and were hard to capture in a photograph. This is in contrast to the waves at the wing tips of Poiseuille flow spots (Carlson *et al.* 1982; Alavyoon *et al.* 1986). From direct observation we estimated the wavelength to be approximately one channel height and the streamwise extent up to ten channel heights. The wave phase velocity was lower than the spreading rate of the spot and the waves were therefore overtaken by the turbulent region (i.e. in a frame of reference relative to the moving border of the spot the waves travelled into the spot). The waves observed at the wing tips of spots in plane Poiseuille flow have been found to reach large amplitudes before breaking down to turbulence and hence play a role in the spanwise spreading of such spots. It is not yet clear if the waves play a similar role in Couette flow.

We were also able to observe waves at *Re* below the transitional *Re* by introducing an obstacle in the channel. These waves were also aligned in the streamwise direction and propagated slowly, with about 4% of U_w, in the spanwise direction.

Finally we mention a completely different instability that was observed when the laminar Couette flow was fully developed and the belt was stopped instantaneously. For this flow almost two-dimensional instabilities developed spanning the height of the channel, with a streamwise wavelength of 13 mm independent of the initial Reynolds number. This may probably be explained by the classical Rayleigh inflexional instability (see for instance Drazin & Reid 1981).

4. Discussion

In the collection of transition data for the canonical flow cases, plane Couette flow has since long been missing. However, the present experiments and the simulations by Lundbladh & Johansson (1991) give results which agree remarkably well concerning the transitional Reynolds number, spot shape and spreading rates. These results also show that available theoretical studies have been unable to predict the transitional Reynolds number.

In the direct numerical simulation of Lundbladh & Johansson (1991) a high-amplitude point-like disturbance is introduced into a laminar Couette flow and its development is followed, either towards a decaying disturbance or towards its transformation into a turbulent spot. They did this calculation for several Reynolds numbers and found growing disturbances and breakdown to a turbulent spot for a $Re = 375$ but not at $Re = 350$. Although computer-time limitations prevent them from following the disturbance for long times their estimate of the transitional Re is in excellent agreement with the present experimental results. The spot shape development, e.g. the original high-aspect-ratio spot which transforms itself into a circular form, was observed both in the simulation and in the present experiments.

The experimental set-up will in the future be used for LDV measurements of the spot structure as well as fully developed turbulent Couette flow. It has been observed from the flow visualization that, at least at low Re, the 'fully' developed turbulent flow seems to be spatially intermittent. Such phenomena are almost impossible to study in other shear flow apparatuses where turbulence propagates in the laboratory frame of reference, but the present set-up is ideal for such a study.

The present apparatus is not only a valuable research tool but it is also an excellent demonstration apparatus to show transition from laminar to turbulent flow, since the centre of a disturbance is fixed in the laboratory frame of reference, making it possible to study the disturbance over long periods of time. The apparatus demonstrates that the disturbance itself is not associated with fluid particles but is a wave phenomenon, illustrating how fluid particles can enter the turbulent region from the laminar flow, take part in the turbulent motion as it moves across the spot and eventually move out of the turbulent region into a laminar flow region.

This study could not have been done without the skilful work of Erik Liljeholm, Department of Gasdynamics, during the planning, construction and manufacturing phases of the Couette flow apparatus. Fruitful discussions with Professors Arne Johansson and Håkan Gustavsson are also acknowledged. An earlier version of this paper was presented at the Third European Turbulence Conference in Stockholm, July 1990. This work was funded by the Swedish National Board for Technical Development through its program for basic research (STUF).

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